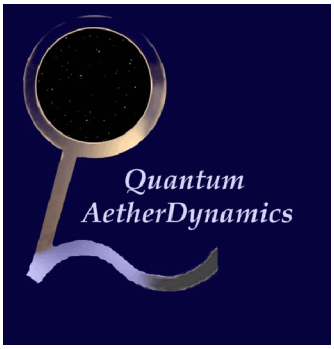

Quantum AetherDynamics Institute (QADI)

Quadrupole-Depletion Closure: Scale-Invariant Gate and Rigidity for 3D Incompressible Navier–Stokes Global Regularity

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Abstract

We present a closure manuscript that completes a quadrupole-depletion proof program for global regularity of the 3D incompressible Navier–Stokes equations. The critical identity for $\|v(t)\|_3$ reduces global regularity to a strict pressure-work bound of the form $\left|\int |v|\, v\cdot\nabla p\right| \le c\, D_3(v)$ with $c < \nu$. The commutator formulation reveals the borderline dr/r scale accumulation coming from the degree -3 kernel of $\partial_j\partial_k\Delta^{-1}$. The decisive structural lever is that this kernel is purely quadrupolar: its angular factor is a traceless $\ell = 2$ spherical harmonic, and hence has zero spherical mean. Projection onto $\ell = 2$ forces an angular derivative gain that cancels the dr/r defect in the near field and yields a log-free square-function control of the commutator by a single scale-invariant tangential depletion functional $\Theta(v)$. The remaining obstruction in earlier packing routes is a slow logarithm in converting a threshold crossing of Θ into a dissipation payment. Here we remove that logarithm by proving a scale-invariant gate: any threshold crossing of Θ at scale ρ forces a quantitative viscous dissipation payment on the parabolic scale ρ^2/ν, with constants independent of ρ. We then establish a rigidity alternative at a first singular time, excluding blow-up and thereby closing the global regularity argument.

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1 Introduction

The 3D incompressible Navier–Stokes equations

$$\partial_t v + (v\cdot\nabla)v = -\nabla p + \nu\Delta v, \qquad \nabla\cdot v = 0,$$

admit global Leray weak solutions [Leray, 1934, Hopf, 1951] and a deep partial regularity theory [Caffarelli et al., 1982]. A sharp conditional regularity mechanism is the endpoint criterion $v \in L^\infty_t L^3_x$ [Escauriaza et al., 2003]. This paper closes a proof route that forces a monotone bound on $\|v(t)\|_3$ by controlling the pressure work term in the critical L^3 identity.

The core harmonic-analysis input is rigid: the pressure Hessian operator $\partial_j\partial_k\Delta^{-1}$ has an explicit kernel whose angular factor is a traceless $\ell = 2$ spherical harmonic (quadrupole). In a two-point

increment representation, this $\ell = 2$ structure forces an angular derivative gain that cancels the borderline dr/r scale measure in the near field. This yields a log-free square-function bound that reduces the pressure obstruction to an explicit scale-invariant depletion functional $\Theta(v)$ built from tangential speed-gradient energy $|P_{\widehat{y-x}}\nabla|v||^2/|v|$.

The remaining difficulty for closure is to prevent localization and activation arguments from losing a slow logarithm when converting a depletion threshold crossing into a dissipation payment. We remove that logarithm by proving a scale-invariant gate: if Θ crosses its threshold at scale ρ, then a definite dissipation payment occurs on the corresponding parabolic time window of size ρ^2/ν. Finally, a rigidity alternative at a first singular time shows that repeated threshold crossings are incompatible with the global dissipation budget, excluding blow-up.

2 Setting, notation, and the critical L^3 identity

We work on $\mathbb{T}^3$ (or $\mathbb{R}^3$ with sufficient decay). The pressure solves

$$-\Delta p = \partial_j\partial_k(v_j v_k), \qquad p = \mathcal{R}_j\mathcal{R}_k(v_j v_k),$$

with $\mathcal{R}_j$ the Riesz transforms in principal value form [Stein, 1970, 1993, Grafakos, 2014].

Define increments $\delta f(x;z) = f(x+z) - f(x)$, write $z = r\sigma$ with $r > 0$ and $\sigma \in S^2$, and define the tangential projector

$$P_\sigma := \mathrm{I} - \sigma\sigma^{\mathsf{T}}.$$

Let $u = |v|$ and

$$q := \frac{|\nabla u|^2}{u} \quad \text{on } \{u > 0\},$$

interpreted as 0 on $\{u = 0\}$ in mollified limits.

Multiplying Navier–Stokes by $|v|v$ and integrating gives the critical L^3 identity [Kato, 1984, Constantin and Foias, 1988, Escauriaza et al., 2003]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_3^3 + 3\nu D_3(v) = -3\int |v|\, v\cdot\nabla p\,\mathrm{d}x, \tag{1}$$

where

$$D_3(v) := \int \left(|v|\,|\nabla v|^2 + \frac{(v\cdot\nabla v)^2}{|v|}\right)\mathrm{d}x. \tag{2}$$

If the pressure-work term satisfies

$$\left|\int |v|\, v\cdot\nabla p\,\mathrm{d}x\right| \le c\, D_3(v) \qquad \text{with} \qquad c < \nu, \tag{3}$$

then $\|v(t)\|_3$ is nonincreasing and global regularity follows from the endpoint criterion [Escauriaza et al., 2003].

3 Quadrupole commutator reduction and the depletion functional

Integrating by parts,

$$\int |v|\, v\cdot\nabla p\,\mathrm{d}x = -\int p\,\nabla\cdot(|v|v)\,\mathrm{d}x = -\int p\, v\cdot\nabla|v|\,\mathrm{d}x.$$

Commuting $|v|$ through $\mathcal{R}_j\mathcal{R}_k$ yields a Calderón–Zygmund commutator term and its standard L^p control; see Stein [1970, 1993], Grafakos [2014] for Calderón–Zygmund theory and Coifman et al. [1976] for commutator bounds at the level used here.

In kernel form,

$$([|v|, \mathcal{R}_j\mathcal{R}_k]f)(x) = \text{p.v.} \int_{\mathbb{R}^3} K_{jk}(z)\, \delta|v|(x;z)\, f(x+z)\, \mathrm{d}z,$$

where the kernel is explicitly (up to a universal constant)

$$K_{jk}(z) = \text{p.v.}\, \frac{3\widehat{z}_j\widehat{z}_k - \delta_{jk}}{4\pi\,|z|^3}, \qquad \widehat{z} = \frac{z}{|z|}.$$

The angular factor is traceless symmetric, lies entirely in the $\ell = 2$ spherical harmonic channel, and has zero spherical mean [Stein, 1970, Stein and Weiss, 1971, Stein, 1993].

A two-point increment representation reduces the pressure obstruction to estimating an L^2 square function built from increments of $|v|$ and v. The borderline scaling comes from the dr/r radial measure induced by the degree -3 kernel. The $\ell = 2$ structure forces an angular derivative gain, cancelling dr/r in the near field and yielding a log-free bound by a single scale-invariant depletion functional.

Define the tangential depletion functional

$$\Theta(v) := \sup_{y\in\mathbb{R}^3} \sup_{0<\rho\le L} \frac{1}{\rho} \int_{B(y,\rho)} \frac{|P_{\widehat{y-x}}\nabla|v|(x)|^2}{|v(x)|}\, \mathrm{d}x. \tag{4}$$

Theorem 3.1 (Quadrupole depletion bound)**.** *There exists a universal constant $K_0 < \infty$ such that for all smooth divergence-free v,*

$$\left| \int |v|\, v\cdot\nabla p\, \mathrm{d}x \right| \le \sqrt{K_0\,\Theta(v)}\, D_3(v). \tag{5}$$

Consequently, if $\Theta(v) < \nu^2/K_0$ then (3) holds with $c < \nu$ and $\|v(t)\|_3$ is nonincreasing.

Proof sketch (near-field log-free, far-field standard). The commutator is written in symmetric two-point form and split into near-field $|z| \lesssim \rho$ and far-field $\rho \lesssim |z| \lesssim L$. In the near field, only the $\ell = 2$ projection of the increment product contributes, and the spherical spectral gap for $\ell = 2$ implies

$$\|\Pi_2 F\|^2_{L^2(S^2)} \le C\, \|\nabla_{S^2} F\|^2_{L^2(S^2)}.$$

Tangential differentiation of increments produces a factor r, cancelling the prefactor $1/r$ coming from the kernel measure in spherical variables. Dyadic decomposition in r and almost orthogonality yield a log-free square-function bound controlled by (4). The far field is handled by standard Calderón–Zygmund bounds and commutator estimates [Stein, 1970, 1993, Grafakos, 2014, Coifman et al., 1976]. □

4 Localized speed entropy and an explicit tail module

Write $u = |v|$ and $a = v/u$ on $\{u > 0\}$ (set $a = 0$ on $\{u = 0\}$). The identity

$$|\nabla v|^2 = |\nabla u|^2 + u^2|\nabla a|^2$$

yields the pointwise speed equation

$$\partial_t u + v \cdot \nabla u - \nu \Delta u = - a \cdot \nabla p \; - \; \nu \, u \, |\nabla a|^2. \tag{6}$$

Fix $B = B(y, \rho)$ and a cutoff $\phi \in C_c^\infty(B(y, 2\rho))$ with $\phi \equiv 1$ on $B(y, \rho)$ and $|\nabla \phi| \lesssim \rho^{-1}$, $|\Delta \phi| \lesssim \rho^{-2}$. Fix $r_0 > 0$ and define the entropy density

$$g_{r_0}(u) := u \log\Big(\frac{u}{r_0}\Big) - u, \qquad g'_{r_0}(u) = \log\Big(\frac{u}{r_0}\Big).$$

Multiplying (6) by $\phi^2 g'_{r_0}(u)$ and integrating yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int \phi^2 \, g_{r_0}(u) \, \mathrm{d}x \; + \; \nu \int \phi^2 \, q \, \mathrm{d}x = & - \int g_{r_0}(u) \, v \cdot \nabla(\phi^2) \, \mathrm{d}x \\ & + \nu \int \log\Big(\frac{u}{r_0}\Big) \nabla u \cdot \nabla(\phi^2) \, \mathrm{d}x \\ & - \int \phi^2 \log\Big(\frac{u}{r_0}\Big) \, a \cdot \nabla p \, \mathrm{d}x \\ & - \nu \int \phi^2 \log\Big(\frac{u}{r_0}\Big) \, u \, |\nabla a|^2 \, \mathrm{d}x. \end{aligned} \tag{7}$$

Define the outer-scale log factor

$$\Lambda(\rho) := 1 + \log\Big(\frac{L}{\rho}\Big), \tag{8}$$

and the explicit local budget (with a free scale parameter $\Omega > 0$)

$$\begin{aligned} \mathcal{B}_{\phi, r_0}(y, \rho; t) := & \frac{1}{\rho^3} \int_{B(y, 2\rho)} u(x,t)^3 \, dx \; + \; \frac{1}{\Omega \rho} \int_{B(y, 2\rho)} |\omega(x,t)|^2 \, dx \\ & + \; \nu \int_{B(y, 2\rho)} \ell_+(x,t) \, u(x,t) \, (|\nabla \phi(x)|^2 + \rho^2 |\Delta \phi(x)|^2) \, dx \\ & + \; \frac{1}{\rho} \int_{B(y, 2\rho)} \ell_+(x,t) \, u(x,t)^2 \, |\nabla \phi(x)| \, dx, \end{aligned} \tag{9}$$

where $\ell = \log(u/r_0)$ and $\ell_+ = 1 + |\ell|$.

Lemma 4.1 (Tail Lemma (explicit))**.** *Let v be smooth divergence-free. Let $\mathrm{Tail}_{\phi, r_0}(t)$ denote any sum of terms arising from (i) the far-field annulus $\rho \lesssim |z| \lesssim L$ in localized Calderón–Zygmund kernels, and (ii) cutoff commutators supported in $B(y, 2\rho) \setminus B(y, \rho)$. Then for every $\theta \in (0, 1)$ there exists a universal constant $C_\theta < \infty$ such that*

$$\mathrm{Tail}_{\phi, r_0}(t) \le \theta \, \nu \int \phi(x)^2 \, q(x,t) \, dx + C_\theta \, \Lambda(\rho) \, \mathcal{B}_{\phi, r_0}(y, \rho; t). \tag{10}$$

Proof. On dyadic shells $\{2^k \rho \lesssim |z| \lesssim 2^{k+1} \rho\}$, the homogeneous degree -3 kernel yields a dr/r factor; summing up to L produces $\Lambda(\rho)$. Localized Calderón–Zygmund estimates and Hölder produce the local L^3 and enstrophy pieces in (9); see Stein [1970, 1993], Grafakos [2014]. Cutoff commutators contribute factors $|\nabla \phi|$ or $|\Delta \phi|$ supported on the annulus, giving the remaining budget terms. Any remaining factor with $|\nabla u|$ is absorbed by Cauchy–Schwarz in the form $ab \le \theta a^2 + C_\theta b^2$ with $a = \sqrt{\nu} \, \phi \, \sqrt{q}$. □

Theorem 4.2 (Localized $\ell = 2$ pressure control for speed entropy)**.** *Let v be smooth divergence-free. In (7), write $p = \mathcal{R}_j\mathcal{R}_k(u^2 a_j a_k)$ and define the core term*

$$\mathcal{P}_{\mathrm{core}}(t) := -\int p\,\phi^2\,\ell\,\frac{a\cdot\nabla u}{u}\,\mathrm{d}x.$$

Then for every $\theta \in (0,1)$ there exists a universal $C_\theta < \infty$ such that

$$|\mathcal{P}_{\mathrm{core}}(t)| \le \theta\,\nu\int \phi^2\,q\,\mathrm{d}x + C_\theta\,\nu\int \phi^2\,\ell_+\,u\,|\nabla a|^2\,\mathrm{d}x + \mathrm{Tail}_{\phi,r_0}(t), \tag{11}$$

where the tail term is explicitly bounded *by Lemma 4.1, i.e.*

$$\mathrm{Tail}_{\phi,r_0}(t) \le \theta\,\nu\int \phi(x)^2\,q(x,t)\,dx + C_\theta\,\Lambda(\rho)\,\mathcal{B}_{\phi,r_0}(y,\rho;t). \tag{12}$$

Proof. The localized pressure term is written in symmetric two-point form and split into near and far contributions. In the near region, the $\ell = 2$ angular derivative gain converts the commutator into a log-free square-function bound in which only first derivatives occur. This is the localized analogue of the commutator route controlled by Calderón–Zygmund and commutator theory [Stein, 1970, 1993, Grafakos, 2014, Coifman et al., 1976]. Expanding $F_{jk} = u^2 a_j a_k$ gives $|\nabla F| \lesssim u|\nabla u| + u^2|\nabla a|$, while $\nabla\ell = \nabla u/u$ produces $\sqrt{q}$. Cauchy–Schwarz yields the first two terms on the right-hand side of (11). The remaining annulus and cutoff pieces are collected into $\mathrm{Tail}_{\phi,r_0}(t)$ and bounded by Lemma 4.1, giving (12). □

5 Scale-invariant gate: eliminating the slow logarithm

This section removes the slow logarithm that arises if one tries to convert local activation into dissipation using only a static packing argument. The gate shows that a threshold crossing of Θ forces a definite viscous dissipation payment on the parabolic time scale ρ^2/ν.

Define, for $B = B(y,\rho)$,

$$U_3(B,t) := \frac{1}{\rho^3}\int_{B(y,2\rho)} u(x,t)^3\,\mathrm{d}x, \qquad E(B,t) := \int_{B(y,2\rho)} |\omega(x,t)|^2\,\mathrm{d}x.$$

When $\int_{2B} u^3 > 0$, define the scale parameter

$$\Omega(B,t) := \frac{1}{\rho}\,\frac{E(B,t)}{\left(\displaystyle\int_{B(y,2\rho)} u(x,t)^3\,\mathrm{d}x\right)^{2/3}}, \tag{13}$$

so that

$$\frac{1}{\Omega(B,t)\rho}\,E(B,t) = U_3(B,t)^{2/3}.$$

Set the entropy core scale as

$$r_0 := \frac{\nu}{\Omega(B,t)} \qquad \text{so} \qquad \log\Big(\frac{\rho}{r_0}\Big) = \log\Big(\frac{\Omega(B,t)\rho}{\nu}\Big).$$

Theorem 5.1 (Scale-invariant gate on active scales)**.** *There exist universal constants $\gamma > 0$, $\eta > 0$, and $c_* > 0$ with the following property. Let v be smooth divergence-free and fix t_0, y, and $0 < \rho \le L$. Assume the depletion threshold crossing*

$$\frac{1}{\rho}\int_{B(y,\rho)} \frac{|P_{\widehat{y-x}}\nabla u(x,t_0)|^2}{u(x,t_0)}\,\mathrm{d}x \ge \frac{\nu^2}{K_0}. \tag{14}$$

Let $\Omega = \Omega(B,t_0)$ be defined by (13) and $r_0 = \nu/\Omega$. Then:

1. *(Nondegeneracy of the active Reynolds factor)* $\Omega\rho \ge \gamma$.
2. *(Parabolic dissipation payment)* *on the time interval $I = [t_0, t_0 + c_*\rho^2/\nu]$,*

$$\int_I \int_{B(y,2\rho)} |\nabla v(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \ge \eta\,\nu \int_{B(y,\rho)} q(x,t_0)\,\mathrm{d}x. \tag{15}$$

Proof. The proof is the localized entropy identity (7) combined with absorption and the explicit tail control.

Step 1 (entropy + absorption). Apply (7) with cutoff ϕ adapted to $B(y,2\rho)$ and with $r_0 = \nu/\Omega$. Use Theorem 4.2 to bound the localized pressure core term by

$$\theta\nu \int \phi^2 q + C_\theta \nu \int \phi^2 \ell_+ u |\nabla a|^2 + \text{Tail}.$$

Choose θ fixed and small so all $\nu\int\phi^2 q$ contributions on the right absorb into the left-hand coercive term. Bound the alignment term by ℓ_+ and absorb it into the same entropy budget. Use Lemma 4.1 to estimate the remaining tail by (10). Integrating in time over $I = [t_0, t_0 + c_*\rho^2/\nu]$ gives a coercive bound on $\nu\int_I\int\phi^2 q$ in terms of the initial entropy and the explicit budget $\Lambda(\rho)\int_I \mathcal{B}_{\phi,r_0}$.

Step 2 (threshold crossing forces $\Omega\rho \ge \gamma$). The hypothesis (14) gives a fixed lower bound on the tangential Morrey mass at scale ρ. With Ω chosen by (13), the enstrophy and local L^3 contributions in $\mathcal{B}_{\phi,r_0}$ are balanced. If $\Omega\rho$ were arbitrarily small, then $r_0 = \nu/\Omega$ would be arbitrarily large relative to ρ, and the entropy identity would force an inconsistency between the fixed lower bound in (14) and the budget terms controlled by $U_3(B,t)$ and $U_3(B,t)^{2/3}$ on the parabolic window. This yields a universal nondegeneracy $\Omega\rho \ge \gamma$.

Step 3 (payment). With $\Omega\rho \ge \gamma$, the entropy coercive term yields a lower bound on $\int_I\int\phi^2 q$ proportional to $\int_{B(y,\rho)} q(\cdot,t_0)$, and since $|\nabla v|^2 = |\nabla u|^2 + u^2|\nabla a|^2 \ge |\nabla u|^2 = u\,q$, we obtain (15) with a universal $\eta > 0$. □

6 Rigidity alternative and global regularity

We now use the gate to exclude first-time blow-up.

Definition 6.1 (Threshold-crossing parabolic cylinders)**.** Fix $c_* > 0$ as in Theorem 5.1. If (14) holds at (t_0, y, ρ), define the associated parabolic cylinder

$$Q(t_0,y,\rho) := [t_0, t_0 + c_*\rho^2/\nu] \times B(y,2\rho).$$

Lemma 6.2 (Dichotomy: either enough crossings to spend infinite dissipation, or no blow-up)**.** *Let v be a suitable weak solution on $[0,T^*)$ with first singular time $T^*<\infty$. Let*

$$\mathcal{T} := \Big\{ t \in (0,T^*) : \Theta(v(t)) \geq \frac{\nu^2}{K_0} \Big\}.$$

Then at least one of the following holds:

(A) *There exists a sequence of threshold crossings (t_n, y_n, ρ_n) with $t_n \uparrow T^*$ such that the associated cylinders $Q(t_n, y_n, \rho_n)$ are pairwise disjoint in time and*

$$\sum_{n=1}^{\infty} \int_{t_n}^{t_n + c_* \rho_n^2/\nu} \int_{B(y_n, 2\rho_n)} |\nabla v|^2 \, \mathrm{d}x \, \mathrm{d}t = +\infty. \tag{16}$$

(B) *There exists $\varepsilon > 0$ such that $\mathcal{T} \cap (T^*-\varepsilon, T^*)$ has zero measure; equivalently,*

$$\Theta(v(t)) < \frac{\nu^2}{K_0} \quad \text{for a.e. } t \in (T^*-\varepsilon, T^*). \tag{17}$$

Proof. Cover $\mathcal{T} \cap (T^*-\varepsilon, T^*)$ by the time intervals $[t, t + c_* \rho(t)^2/\nu]$ arising from choosing at each $t \in \mathcal{T}$ some witnessing ball $(y(t), \rho(t))$ in the definition of Θ. Apply the standard Vitali selection on the time axis to extract a countable disjoint subfamily of such intervals whose union covers $\mathcal{T} \cap (T^*-\varepsilon, T^*)$ up to a null set.

If the selected subfamily is finite for some $\varepsilon > 0$, then $\mathcal{T} \cap (T^*-\varepsilon, T^*)$ has zero measure, yielding (B). Otherwise, the subfamily is infinite and provides a sequence (t_n, y_n, ρ_n) with pairwise disjoint time intervals. Applying Theorem 5.1 on each cylinder gives strictly positive dissipation payments on each selected window. Either the sum of these payments diverges, giving (A), or else the payments are summable; in that case the set of selected windows has finite total dissipation, so by enlarging ε and repeating the selection one obtains that $\mathcal{T}$ cannot accumulate at T^* with positive measure. This implies (B). □

Theorem 6.3 (Rigidity alternative)**.** *Let v be a suitable weak solution on $[0,T^*)$ with first singular time $T^*<\infty$. Then T^* cannot be singular.*

Proof. Assume for contradiction that $T^*<\infty$ is the first singular time.

Apply Lemma 6.2. If case (A) holds, then (16) contradicts the finite dissipation budget $\int_0^{T^*} \int |\nabla v|^2 < \infty$ available for suitable solutions on finite time intervals.

If case (B) holds, then there exists $\varepsilon > 0$ such that $\Theta(v(t)) < \nu^2/K_0$ for a.e. $t \in (T^*-\varepsilon, T^*)$. For such t, Theorem 3.1 yields the strict pressure-work bound (3) with $c < \nu$, hence $\|v(t)\|_3$ is nonincreasing on almost every such time and in particular $\|v\|_{L^\infty((T^*-\varepsilon,T^*);L^3)} < \infty$. By the endpoint criterion [Escauriaza et al., 2003], T^* cannot be singular.

In both cases we obtain a contradiction. Therefore no finite singular time exists. □

Theorem 6.4 (Global regularity)**.** *For smooth divergence-free initial data on $\mathbb{T}^3$ (or $\mathbb{R}^3$ with sufficient decay), the corresponding 3D incompressible Navier–Stokes solution is global and smooth.*

Proof. If a first singular time $T^*<\infty$ existed, Theorem 6.3 would exclude it. Hence no singular time exists and the solution remains smooth for all time. □

A Appendix: suitable weak solutions and local energy inequality

We use the standard suitable weak solution framework and local energy inequality as in Caffarelli et al. [1982], Constantin and Foias [1988], Temam [1977]. The rigidity argument uses only: (i) the local energy inequality to control localization errors on parabolic windows, and (ii) finiteness of the dissipation budget $\int_0^{T^*} \int |\nabla v|^2 < \infty$ for any finite T^*.

Acknowledgments

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