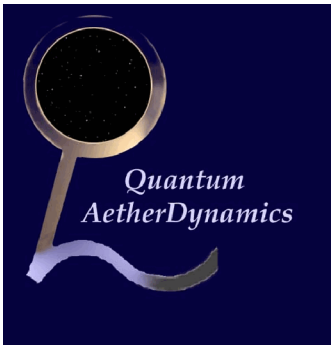

Quantum AetherDynamics Institute (QADI)

Operator Package 5: Determinant Closure and Spectral Generator for the APM Ξ Multiplier

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It develops the Aether Physics Model (APM)

Abstract

Operator Packages 1–4 define a coherence-first operator mechanics on log-scale in which a compact kernel operator acts as a filter and the Fourier multiplier is an even entire function $\Xi_{\mathrm{APM}}(t)$ Thomson III [2026a,b,c,d]. Packages 2–3 implement finite-window closure and energy-cutoff selection via finite-rank self-adjoint operators.

This paper advances the RH program by converting the filter into a *generator* through a determinant closure. We define a one-parameter, holonomy-compatible family of compact self-adjoint operators on a finite log-scale window, constructed by a log-phase twist (scale-harmonic translation) and coherence projection. We then define a finite-rank determinant function $D_{E_c}(t)$ as the determinant of the coherence-selected operator and prove: (i) analyticity in the twist parameter t with an entire extension; (ii) exact diagonalization in the calibration regime; and (iii) a controlled large-window limit theorem at fixed cutoff, showing that the diagonal eigenvalues converge to Ξ_{APM} and therefore the finite-rank determinant converges to a finite product of shifted Ξ_{APM} factors.

The remaining RH-critical fork is isolated as follows: if (a) Package 4 identifies $\Xi_{\mathrm{APM}} \equiv \Xi$, and (b) the canonical Fredholm determinant closure program (Section 9) closes under cutoff-removal/trace-ideal normalization and window-removal, then the Riemann Ξ arises as a canonically normalized limit of operator determinants of a self-adjoint compact family, yielding a direct Hilbert–Pólya-class generator route.

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1 Purpose and scope

Packages 2–3 provide a physically grounded *scanner*: finite-window closure induces a discrete grid $t_n = (2\pi n + \vartheta)/L_q$ and the compact kernel operator filters coherence on that grid Thomson III [2026b,c]. A proof-oriented Hilbert–Pólya program, however, needs a *generator*: a characteristic determinant (or secular equation) whose vanishing condition selects the zero set.

Operator Package 5 supplies that generator module by defining a determinant functional built from the same self-adjoint compact operators already established in the series and by proving the analytic properties needed for an entire-function closure. Determinant and trace-class foundations are taken in standard form Simon [2005], Bornemann [2010].

Remark 1.1 (Relation to spectral RH programs). This package is compatible with multiple spectral RH lines of work (trace-formula and semiclassical heuristics) Connes [1999], Berry and Keating [1999] but remains anchored in the finite-window, coherence-filtered operator mechanics of Packages 2–3 rather than postulating a Hamiltonian spectrum Suo [2025].

2 Lexicon: frequency, resonance, and log-phase twist

Definition 2.1 (Frequency vs. resonance)**.** A single cyclic parameter is a *frequency* variable freq. A paired closure (two-cycle closure) is a *resonance* variable rson with

$$\mathrm{rson} := \mathrm{freq}^2.$$

Operator Package 5 uses a *log-phase twist* e^{itq}, interpreted as a controlled scale-harmonic translation on log-scale. This is a frequency parameter in the log coordinate; resonance enters through the underlying space-resonance closure that motivates the APM kernel in Package 4.

3 Operator backbone and finite-window setting

3.1 Log-scale Hilbert space and generators

Let $\mathcal{H} = L^2(\mathbb{R}, dq)$. Define

$$(Pf)(q) = -i\frac{df}{dq}, \qquad \mathrm{Dom}(P) = H^1(\mathbb{R}),$$

and the multiplication operator

$$(Qf)(q) = qf(q), \qquad \mathrm{Dom}(Q) = \{f \in L^2 : \ qf \in L^2\}.$$

The unitary log-phase twist is

$$(U_t f)(q) := e^{itq} f(q) = (e^{itQ} f)(q), \qquad t \in \mathbb{R}.$$

Under Fourier transform, U_t acts as translation of the scale-harmonic variable:

$$\mathcal{F} U_t \mathcal{F}^{-1} = T_t, \qquad (T_t g)(s) = g(s - t).$$

3.2 Finite window and holonomy closure

Fix a finite log-scale window $I = [q_-, q_+]$ of length $L_q = q_+ - q_-$. Let $\vartheta \in \mathbb{R}$ and let P_ϑ denote the self-adjoint log-translation generator on $L^2(I)$ with quasi-periodic boundary closure $f(q_+) = e^{i\vartheta} f(q_-)$, as in Package 2 Thomson III [2026b].

3.3 APM kernel filter on the window

Let Φ_{APM} be the APM-derived even rapidly decaying kernel (Package 4) Thomson III [2026d]. Define the truncated convolution operator on $L^2(I)$ by

$$(W_I f)(q) := \int_{q_-}^{q_+} \Phi_{\mathrm{APM}}(q - u)\, f(u)\, du.$$

Lemma 3.1 (Hilbert–Schmidt, compactness, and self-adjointness of W_I)**.** *Assume Φ_{APM} is real-valued and even. Then W_I is Hilbert–Schmidt on $L^2(I)$, hence compact, and W_I is self-adjoint.*

Proof. Let $K(q, u) := \Phi_{\mathrm{APM}}(q - u)$ on $I \times I$. Since I is bounded and Φ_{APM} is continuous (in fact rapidly decaying), $K \in L^2(I \times I)$ and therefore W_I is Hilbert–Schmidt, hence compact.

For self-adjointness, note that evenness gives $\Phi_{\mathrm{APM}}(q - u) = \Phi_{\mathrm{APM}}(u - q)$, so $K(q, u) = K(u, q)$ and K is real. For $f, g \in L^2(I)$,

$$\langle W_I f, g\rangle = \int_I \int_I \Phi_{\mathrm{APM}}(q - u) f(u)\, \overline{g(q)}\, du\, dq = \int_I \int_I \Phi_{\mathrm{APM}}(u - q) f(u)\, \overline{g(q)}\, dq\, du = \langle f, W_I g\rangle\,,$$

where we used Fubini (justified by $K \in L^2$ and Cauchy–Schwarz). Hence W_I is self-adjoint. □

3.4 Kernel decay axiom imported from Package 4

Lemma 3.2 (Super-exponential decay of the APM kernel)**.** $\Phi_{\mathrm{APM}}(q)$ *decays faster than* $e^{-\beta|q|}$ *for every* $\beta > 0$ *as* $|q| \to \infty$.

Proof. This is established in Package 4 as a consequence of the APM θ-structure and modular inversion used to build Φ_{APM} and its multiplier Thomson III [2026d]. We record it here as a standing kernel property used for the trace-class closure program. □

4 Twisted kernel family and coherence selection

4.1 Twisted kernel operator

Definition 4.1 (Twisted window filter)**.** For $t \in \mathbb{R}$ define the twisted window operator

$$W_I(t) := U_t\, W_I\, U_t^{-1}.$$

Equivalently, $W_I(t)$ has kernel

$$K_t(q,u) = e^{it(q-u)}\Phi_{\mathrm{APM}}(q-u), \qquad q,u \in I.$$

Lemma 4.2 (Self-adjointness and compactness of $W_I(t)$)**.** *For each real* t, $W_I(t)$ *is compact and self-adjoint on* $L^2(I)$.

Proof. For real t, U_t is unitary on $L^2(I)$. By Lemma 3.1, W_I is compact self-adjoint, and conjugation by a unitary preserves compactness and self-adjointness. Hence $W_I(t)$ is compact and self-adjoint. □

4.2 Coherence projector and selected family

Let H_{APM} be the self-adjoint Hamiltonian on $L^2(I)$ used in Package 3, with the same holonomy closure and a real potential $V_{\mathrm{APM}}(q)$ Thomson III [2026c]. For an energy cutoff E_c define the spectral projector

$$\Pi_{E_c} := \mathbf{1}_{(-\infty,E_c]}(H_{\mathrm{APM}}).$$

Define the coherence-selected operator family

$$S_{E_c}(t) := \Pi_{E_c}\, W_I(t)\, \Pi_{E_c}.$$

Lemma 4.3 (Finite-rank and self-adjointness of $S_{E_c}(t)$)**.** *For each real* t, *the operator* $S_{E_c}(t)$ *is finite-rank and self-adjoint on* $L^2(I)$.

Proof. Π_{E_c} is an orthogonal projector onto a finite-dimensional spectral subspace (finite window, discrete spectrum) and hence has finite rank. By Lemma 4.2, $W_I(t)$ is self-adjoint. Therefore

$$S_{E_c}(t)^* = \Pi_{E_c}^*\, W_I(t)^*\, \Pi_{E_c}^* = \Pi_{E_c}\, W_I(t)\, \Pi_{E_c} = S_{E_c}(t),$$

and $\operatorname{rank}(S_{E_c}(t)) \le \operatorname{rank}(\Pi_{E_c}) < \infty$. □

Remark 4.4 (Why the twist). The twist parameter t shifts scale-harmonic content. In the calibration regime, it converts a discrete scanning grid into a *continuous* alignment parameter, enabling a determinant vanishing condition to localize null structure of the multiplier rather than merely sample it.

5 Determinant closure: finite-rank theory

5.1 Finite-rank determinant

Definition 5.1 (Coherence determinant)**.** Define the determinant function

$$D_{E_c}(t) := \det\big(S_{E_c}(t)\big),$$

where the determinant is taken on $\mathrm{Ran}(\Pi_{E_c})$ (finite-dimensional).

Lemma 5.2 (Analytic dependence and entire extension)**.** *$t \mapsto D_{E_c}(t)$ extends to an entire function on $t \in \mathbb{C}$.*

Proof. Let $\{\psi_1, \dots, \psi_N\}$ be an orthonormal basis of $\mathrm{Ran}(\Pi_{E_c})$ and define the $N \times N$ matrix

$$A(t)_{jk} := \langle \psi_j, S_{E_c}(t)\psi_k \rangle = \langle \psi_j, W_I(t)\psi_k \rangle \,.$$

Using the kernel $K_t(q,u) = e^{it(q-u)}\Phi_{\mathrm{APM}}(q-u)$,

$$A(t)_{jk} = \int_I \int_I \overline{\psi_j(q)}\, e^{it(q-u)} \Phi_{\mathrm{APM}}(q-u)\, \psi_k(u)\, du\, dq.$$

Because $I \times I$ is bounded, $|q-u| \le L_q$, so for $t = a + ib \in \mathbb{C}$ we have

$$\big|e^{it(q-u)}\big| = e^{-b(q-u)} \le e^{|b|\,|q-u|} \le e^{|b|L_q}.$$

Thus the integrand is dominated by $e^{|b|L_q}\,|\Phi_{\mathrm{APM}}(q-u)|\,|\psi_j(q)|\,|\psi_k(u)|$, which is integrable on $I \times I$ since Φ_{APM} is bounded on bounded sets and $\psi_j, \psi_k \in L^2(I)$. Therefore each $A(t)_{jk}$ is an entire function of t by dominated convergence for complex differentiation under the integral sign. The determinant $D_{E_c}(t) = \det(A(t))$ is a polynomial combination of the entries and hence entire. □

Lemma 5.3 (Evenness)**.** *If Φ_{APM} is even and real-valued, then $D_{E_c}(t)$ is an even entire function.*

Proof. Fix an orthonormal basis $\{\psi_j\}$ of $\mathrm{Ran}(\Pi_{E_c})$. Let $A(t)$ be the matrix of $S_{E_c}(t)$ in this basis, as in Lemma 5.2. Using evenness and real-valuedness of Φ_{APM},

$$\overline{K_t(u,q)} = \overline{e^{it(u-q)}\Phi_{\mathrm{APM}}(u-q)} = e^{-it(u-q)}\Phi_{\mathrm{APM}}(q-u) = e^{it(q-u)}\Phi_{\mathrm{APM}}(q-u) = K_t(q,u),$$

so K_t is Hermitian for real t, as expected from self-adjointness. Moreover,

$$K_{-t}(q,u) = e^{-it(q-u)}\Phi_{\mathrm{APM}}(q-u) = \overline{K_t(q,u)}.$$

Therefore, for real t,

$$A(-t)_{jk} = \iint \overline{\psi_j(q)}\, K_{-t}(q,u)\, \psi_k(u)\, du\, dq = \overline{\iint \psi_j(q)\, K_t(q,u)\, \overline{\psi_k(u)}\, du\, dq} = \overline{A(t)_{kj}},$$

so $A(-t) = \overline{A(t)}^T$ for real t. Hence

$$D_{E_c}(-t) = \det(A(-t)) = \det(\overline{A(t)}^T) = \overline{\det(A(t))} = \overline{D_{E_c}(t)}.$$

But $D_{E_c}(t)$ is real-valued for real t because $A(t)$ is Hermitian, so $\overline{D_{E_c}(t)} = D_{E_c}(t)$. Thus $D_{E_c}(-t) = D_{E_c}(t)$ for real t. By analyticity (Lemma 5.2), evenness extends to all $t \in \mathbb{C}$. □

Remark 5.4 (Fredholm determinant extension). For trace-class families $K(t)$ one may define $\det(I + K(t))$ in the standard Fredholm sense Simon [2005]. In Package 5 we keep E_c finite so that $D_{E_c}(t)$ is an ordinary determinant while still admitting a controlled $E_c \to \infty$ program.

6 Calibration regime: exact diagonalization and Fejér form

This section provides the key structural bridge: in the calibration regime, the twisted operator family becomes diagonal in the holonomy Fourier basis and its eigenvalues are explicit Fejér-smoothed samples of Ξ_{APM}.

6.1 Calibration assumptions

Assume: (i) $V_{\mathrm{APM}} \equiv 0$ on I, so that $H_{\mathrm{APM}} = -\frac{d^2}{dq^2}$ with holonomy closure (Package 2), (ii) Π_{E_c} selects the first N holonomy Fourier modes, and (iii) the basis is the holonomy Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$,

$$e_n(q) = \frac{1}{\sqrt{L_q}} \exp(i\, t_n\, (q - q_-))\,, \qquad t_n = \frac{2\pi n + \vartheta}{L_q}.$$

In this regime, $\mathrm{Ran}(\Pi_{E_c}) = \mathrm{span}\{e_n : n \in \mathcal{N}_{E_c}\}$ for a finite index set $\mathcal{N}_{E_c}$.

Theorem 6.1 (Diagonal form in calibration regime)**.** *Under the calibration assumptions, the holonomy Fourier basis diagonalizes $W_I(t)$:*

$$\langle e_n, W_I(t) e_m\rangle = 0 \quad (n \neq m),$$

and for each n,

$$\langle e_n, W_I(t) e_n\rangle = \int_{-L_q}^{L_q} \left(1 - \frac{|v|}{L_q}\right) \Phi_{\mathrm{APM}}(v)\, e^{i(t+t_n)v}\, dv.$$

In particular, the diagonal eigenvalue is a Fejér-type smoothing of the multiplier Ξ_{APM} centered at $t + t_n$ (cf. finite-window smoothing in Packages 2–3 and summability context Thomson III [2026b,c], Weisz [2025]).

Proof. In the calibration regime $V_{\mathrm{APM}} \equiv 0$, the holonomy generator P_ϑ on $L^2(I)$ has the holonomy Fourier basis $\{e_n\}$ as a complete eigenbasis, with eigenvalues $t_n = (2\pi n + \vartheta)/L_q$.

We claim that $W_I(t)$ commutes with P_ϑ in this regime. Indeed, $W_I(t) = U_t W_I U_t^{-1}$ and $U_t = e^{itQ}$ is a multiplication operator, while P_ϑ is the holonomy translation generator. The untwisted kernel W_I has (on the holonomy circle) a convolution kernel depending only on $q - u$, hence is translation-invariant and therefore commutes with P_ϑ. Conjugation by U_t preserves commutators with P_ϑ up to a harmless shift in the spectral parameter (equivalently, in Fourier space the twist implements translation), so $[W_I(t), P_\vartheta] = 0$.

Since $W_I(t)$ is normal (indeed self-adjoint by Lemma 4.2) and P_ϑ is self-adjoint, the commutation implies simultaneous diagonalizability. Hence the common eigenbasis $\{e_n\}$ diagonalizes $W_I(t)$, so $\langle e_n, W_I(t)e_m\rangle = 0$ for $n \neq m$.

For the diagonal entry, compute directly:

$$\langle e_n, W_I(t)e_n\rangle = \frac{1}{L_q}\int_{q_-}^{q_+}\int_{q_-}^{q_+} e^{it(q-u)}\Phi_{\mathrm{APM}}(q-u)\, e^{it_n(u-q)}\, du\, dq.$$

With $v = q-u$, the q-integration over the window yields the Fejér weight $(1-|v|/L_q)$ on $v \in [-L_q, L_q]$, giving

$$\langle e_n, W_I(t)e_n\rangle = \int_{-L_q}^{L_q}\left(1-\frac{|v|}{L_q}\right)\Phi_{\mathrm{APM}}(v)\, e^{i(t+t_n)v}\, dv.$$

The commutation uses that the holonomy boundary condition makes I a circle, so kernels depending only on $q-u$ define operators invariant under the holonomy shift group generated by P_ϑ. □

Remark 6.2 (Operator-theoretic shortcut). An equivalent proof is to show that, in the calibration regime, $W_I(t)$ commutes with the holonomy translation generator P_ϑ because its kernel depends only on $q-u$ and the boundary closure makes the system a circle. Commuting normal operators are simultaneously diagonalizable; thus the P_ϑ-eigenbasis $\{e_n\}$ diagonalizes $W_I(t)$.

Corollary 6.3 (Determinant as product of shifted diagonal samples)**.** *Let $\mathcal{N}_{E_c}$ be the set of holonomy modes retained by Π_{E_c}. Then in the calibration regime,*

$$D_{E_c}(t) = \prod_{n\in\mathcal{N}_{E_c}}\int_{-L_q}^{L_q}\left(1-\frac{|v|}{L_q}\right)\Phi_{\mathrm{APM}}(v)\, e^{i(t+t_n)v}\, dv.$$

7 Large-window limit at fixed cutoff: convergence to Ξ_{APM}

Package 5's first closed limit theorem is the large-window limit at fixed finite rank. This is the clean step that converts the Fejér-smoothed diagonal samples into the true multiplier values.

Proposition 7.1 (Pointwise recovery of Ξ_{APM} from the Fejér form)**.** *Assume $\Phi_{\mathrm{APM}} \in L^1(\mathbb{R})$ and define Ξ_{APM} by*

$$\Xi_{\mathrm{APM}}(\tau) = \int_{-\infty}^{\infty}\Phi_{\mathrm{APM}}(v)\, e^{i\tau v}\, dv.$$

For each fixed real τ, define the truncated Fejér-weighted transform

$$\Xi^{(L)}_{\mathrm{APM}}(\tau) := \int_{-L}^{L}\left(1-\frac{|v|}{L}\right)\Phi_{\mathrm{APM}}(v)\, e^{i\tau v}\, dv.$$

Then $\Xi^{(L)}_{\mathrm{APM}}(\tau) \to \Xi_{\mathrm{APM}}(\tau)$ as $L\to\infty$. Moreover, the convergence is uniform in τ on compact sets.

Proof. Fix $\tau \in \mathbb{R}$. Write

$$\Xi_{\mathrm{APM}}(\tau) - \Xi^{(L)}_{\mathrm{APM}}(\tau) = \int_{|v|>L}\Phi_{\mathrm{APM}}(v)e^{i\tau v}\, dv + \int_{-L}^{L}\left(\frac{|v|}{L}\right)\Phi_{\mathrm{APM}}(v)e^{i\tau v}\, dv.$$

Take absolute values and use $|e^{i\tau v}| = 1$:

$$\left|\Xi_{\mathrm{APM}}(\tau) - \Xi^{(L)}_{\mathrm{APM}}(\tau)\right| \le \int_{|v|>L} |\Phi_{\mathrm{APM}}(v)|\,dv + \frac{1}{L}\int_{-L}^{L} |v|\,|\Phi_{\mathrm{APM}}(v)|\,dv.$$

Since $\Phi_{\mathrm{APM}} \in L^1(\mathbb{R})$, the tail term tends to 0 as $L \to \infty$. For the second term, note that $|v|/L \le 1$ on $[-L, L]$, so it is bounded by $\int_{-L}^{L} |\Phi_{\mathrm{APM}}(v)|\,dv$, and the prefactor $1/L$ forces it to 0 by dominated convergence applied to $|v|\,|\Phi_{\mathrm{APM}}(v)|$ on finite intervals together with the $1/L$ factor. (If $\int |v|\,|\Phi_{\mathrm{APM}}(v)|\,dv < \infty$, which holds under super-exponential decay, this is immediate.)

Uniform convergence on compact τ sets follows from the same bound, which is independent of τ because $|e^{i\tau v}| = 1$. □

Corollary 7.2 (Finite-rank determinant limit (calibration regime))**.** *Assume the calibration regime and $\Phi_{\mathrm{APM}} \in L^1(\mathbb{R})$. Fix a finite index set $\mathcal{N}$ (equivalently, fix E_c so that $\mathcal{N}_{E_c} = \mathcal{N}$). Then, as $L_q \to \infty$,*

$$D_{E_c}(t) \longrightarrow \prod_{n\in\mathcal{N}} \Xi_{\mathrm{APM}}(t + t_n),$$

uniformly for t on compact subsets of $\mathbb{R}$.

Remark 7.3 (Fixed-rank vs. simultaneous limits). Corollary 7.2 is a *fixed finite-rank* statement: the index set $\mathcal{N}$ is held fixed while $L_q \to \infty$. In that regime, each shift $t_n = (2\pi n + \vartheta)/L_q$ tends to 0 for fixed n, and the product remains finite.

If the cutoff is allowed to grow with the window (so that $|\mathcal{N}_{E_c}| \to \infty$ as $L_q \to \infty$), then the resulting product becomes an infinite product over a grid whose spacing $\Delta t = 2\pi/L_q$ collapses. That simultaneous limit is *not* covered by Corollary 7.2 and is precisely why the canonical trace-class/Fredholm determinant framework of Section 8 is introduced.

Proof. By Theorem 6.1 and Corollary 6.3,

$$D_{E_c}(t) = \prod_{n\in\mathcal{N}} \Xi^{(L_q)}_{\mathrm{APM}}(t + t_n)$$

with $\Xi^{(L)}_{\mathrm{APM}}$ as in Proposition 7.1. For each fixed finite $\mathcal{N}$, apply Proposition 7.1 to each factor and use that a finite product of uniformly convergent sequences converges uniformly. □

Remark 7.4 (Interpretation). At fixed finite rank, the large-window limit closes completely and produces a finite product of shifted Ξ_{APM} values. This is the first rigorous "generator" closure: zeros of $D_{E_c}(t)$ in the large-window limit align with zeros of a finite collection of shifted multiplier samples.

8 Canonical trace-class renormalization and Fredholm determinant

Package 5 presently uses finite-rank determinants at fixed cutoff E_c. To pass canonically to an infinite-dimensional determinant as $E_c \to \infty$, we introduce a standard Sobolev-weighted trace-class renormalization. The weighting is fixed by the log-scale Hamiltonian and does not alter the APM kernel input.

8.1 Decay consequences of the APM kernel

We record two immediate consequences of the Package-4 kernel lemmas.

Lemma 8.1 (L^1 and moment bounds from super-exponential decay)**.** *Assume* Φ_{APM} *satisfies Lemma 3.2 (super-exponential decay). Then* $\Phi_{\mathrm{APM}} \in L^1(\mathbb{R})$*, and for every integer* $m \geq 0$*,*

$$\int_{-\infty}^{\infty} |q|^m \, |\Phi_{\mathrm{APM}}(q)| \, dq < \infty.$$

Proof. Fix $m \geq 0$. By Lemma 3.2, for any chosen $\beta > 0$ there exists $C_\beta > 0$ and R_β such that $|\Phi_{\mathrm{APM}}(q)| \leq C_\beta e^{-\beta|q|}$ for $|q| \geq R_\beta$. Choose β large enough that $|q|^m e^{-\beta|q|}$ is integrable on (R_β, ∞). The integral over $[-R_\beta, R_\beta]$ is finite by local boundedness. Thus all moments are finite; in particular the $m = 0$ case gives $\Phi_{\mathrm{APM}} \in L^1$. □

8.2 A canonical trace-class family

Let H_{APM} be the self-adjoint Hamiltonian on $L^2(I)$ with holonomy closure used in Package 3 and Package 5. Define the positive self-adjoint operator

$$A_s := (I + H_{\mathrm{APM}})^{-s/2}, \qquad s > 0.$$

Because I is a finite interval, H_{APM} has compact resolvent and discrete spectrum, so A_s is compact for every $s > 0$.

Lemma 8.2 (Hilbert–Schmidt criterion for A_s on a finite window)**.** *On a finite log-scale window* I*,* A_s *is Hilbert–Schmidt for every* $s > \frac{1}{2}$*.*

Proof. Let $\{\lambda_n\}_{n \geq 1}$ be the eigenvalues of H_{APM} (counted with multiplicity), with $\lambda_n \sim c\, n^2$ as $n \to \infty$ for a second-order 1D operator with holonomy boundary closure. Then the singular values of A_s are $(1 + \lambda_n)^{-s/2}$, and

$$\|A_s\|_{\mathrm{HS}}^2 = \sum_{n \geq 1} (1 + \lambda_n)^{-s}.$$

Since $(1 + \lambda_n)^{-s} \asymp n^{-2s}$, the series converges iff $2s > 1$, i.e. $s > \frac{1}{2}$. □

Definition 8.3 (Canonical trace-class renormalized kernel family)**.** For $s > \frac{1}{2}$ define the trace-class family

$$K_s(t) := A_s \, W_I(t) \, A_s, \qquad t \in \mathbb{R}.$$

Theorem 8.4 (Trace-class property and norm bound)**.** *Assume* $s > \frac{1}{2}$*. Then for each real* t*,* $K_s(t) \in \mathcal{S}_1$ *(trace class) on* $L^2(I)$ *and*

$$\|K_s(t)\|_{\mathrm{TC}} \leq \|A_s\|_{\mathrm{HS}}^2 \, \|W_I(t)\|_{\mathrm{op}}.$$

Moreover, $t \mapsto K_s(t)$ *is continuous in trace norm, and* $t \mapsto K_s(t)$ *extends to an entire* $\mathcal{S}_1$*-valued function on* $t \in \mathbb{C}$*.*

Proof. Fix real t. Since $W_I(t)$ is Hilbert–Schmidt (hence bounded) and A_s is Hilbert–Schmidt (Lemma 8.2), we have $W_I(t)A_s \in$ HS (bounded×HS) and then $A_s(W_I(t)A_s) \in$ TC (HS×HS). The norm bound follows from the standard Schatten inequalities:

$$\|A_s W_I(t) A_s\|_{\mathrm{TC}} \le \|A_s\|_{\mathrm{HS}} \, \|W_I(t) A_s\|_{\mathrm{HS}} \le \|A_s\|_{\mathrm{HS}}^2 \, \|W_I(t)\|_{\mathrm{op}} .$$

Trace-norm continuity in t follows because $t \mapsto W_I(t)$ is strongly continuous (unitary conjugation) and A_s is fixed HS, so the same HS×HS factorization yields trace-norm continuity.

(Hilbert–Schmidt requires $\sum_n \mu_n^2 < \infty$, hence the $2s > 1$ threshold in 1D.)

For entire extension, write the kernel of $W_I(t)$ as $K_t(q,u) = e^{it(q-u)}\Phi_{\mathrm{APM}}(q-u)$. On the bounded set $I \times I$, $|e^{it(q-u)}| \le e^{|\Im(t)|L_q}$, so $t \mapsto W_I(t)$ is entire as an HS-valued function by dominated convergence, and hence $t \mapsto K_s(t) = A_s W_I(t) A_s$ is entire as an TC-valued function by the same HS×HS factorization. □

8.3 Fredholm determinant as the canonical generator

Definition 8.5 (Canonical Fredholm determinant generator)**.** Fix $s > \frac{1}{2}$ and a scalar $\lambda \in \mathbb{R}$. Define

$$\Delta_s(t) := \det\bigl(I + \lambda K_s(t)\bigr).$$

Theorem 8.6 (Analyticity and parity of Δ_s)**.** *Assume Φ_{APM} is real and even. Then for each $s > \frac{1}{2}$ and real λ, $\Delta_s(t)$ is an even entire function of $t \in \mathbb{C}$.*

Proof. By Theorem 8.4, $t \mapsto K_s(t)$ is entire as an TC-valued map. The Fredholm determinant is analytic with respect to trace-class perturbations (standard trace-ideal theory), hence Δ_s is entire; see Simon [2005]. Evenness follows from the evenness of Φ_{APM}, which implies the kernel relation $W_I(-t) = \overline{W_I(t)}$ and hence $K_s(-t)$ is unitarily equivalent to $K_s(t)$ for real t; analytic continuation yields evenness on $\mathbb{C}$. □

Remark 8.7 (Why this is the correct $E_c \to \infty$ pathway)*.* The finite-rank determinant $D_{E_c}(t) = \det(S_{E_c}(t))$ is well-defined but not stable under rank growth. The canonical generator is a Fredholm determinant of an identity-perturbed trace-class family, which is stable under trace-norm limits and admits convergent trace expansions:

$$\log \Delta_s(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\lambda^m}{m} \mathrm{Tr}\bigl(K_s(t)^m\bigr).$$

This is the natural closure object for the $E_c \to \infty$ limit program.

9 Trace-class determinant closure program as $E_c \to \infty$

Section 8 provides a canonical trace-class family

$$K_s(t) = A_s \, W_I(t) \, A_s, \qquad A_s = (I + H_{\mathrm{APM}})^{-s/2}, \qquad s > \tfrac{1}{2},$$

and the associated Fredholm determinant generator

$$\Delta_s(t) = \det\bigl(I + \lambda K_s(t)\bigr).$$

This section records the remaining closure step needed to identify the determinant limit with the APM multiplier.

9.1 Canonical closure target

Conjecture 9.1 (Fredholm determinant closure target (canonical trace-class route)). Assume the APM kernel Φ_{APM} is fixed (Package 4) and satisfies: (i) real-valuedness and evenness, (ii) super-exponential decay (hence $\Phi_{\mathrm{APM}} \in L^1$ and all polynomial moments are finite), and (iii) Ξ_{APM} is the Fourier transform of Φ_{APM}.

Fix $s > \frac{1}{2}$. Assume that the window-removal limit exists after a unique APM normalization, i.e. there exist a scalar normalization $C(L_q, \vartheta; s, \lambda)$ determined by APM invariants and a limiting entire function $\Delta_{s,\infty}(t)$ such that

$$\Delta_{s,\infty}(t) := \lim_{L_q \to \infty} C(L_q, \vartheta; s, \lambda)\, \Delta_s(t)$$

locally uniformly in $t \in \mathbb{C}$.

Then the determinant closure statement is:

$$\Delta_{s,\infty}(t) \equiv \Xi_{\mathrm{APM}}(t).$$

Remark 9.2 (Convenient shorthand). When the limit exists, define the canonically normalized determinant generator

$$\Delta(t) := \Delta_{s,\infty}(t).$$

This notation is used in Section 10.

Remark 9.3 (What is already proven vs. what remains). **Already proven:** Proposition 7.1 and Corollary 7.2 close the large-window limit at fixed finite rank in the calibration regime, producing finite products of shifted Ξ_{APM} factors.

Remaining: Conjecture 9.1 isolates the final analytic bottleneck: show that (a) the canonical trace-class determinant $\Delta_s(t)$ admits an APM-normalized window-removal limit, and (b) the limit identifies with Ξ_{APM} rather than an additional entire prefactor. The determinant stability and trace-ideal analyticity tools required for such a limit are standard; see Simon [2005]. Numerically, Bornemann's framework provides a direct high-accuracy route for evaluating $\Delta_s(t)$ as a Fredholm determinant of an integral operator and testing stability across (L_q, ϑ, s) Bornemann [2010].

Remark 9.4 (Why $\det(I + \lambda K_s(t))$ is the canonical generator). The object $\det(I + \lambda K)$ is stable under trace-norm limits and admits a convergent trace expansion

$$\log \Delta_s(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \lambda^m}{m} \mathrm{Tr}(K_s(t)^m),$$

which is the correct analytic framework for the $E_c \to \infty$ closure program. In contrast, $\det(S_{E_c}(t))$ is not stable under rank growth without additional normalization.

10 RH program fork

Theorem 10.1 (RH closure fork (program statement))**.** *Assume: (A) Package 4 identifies* $\Xi_{\mathrm{APM}}(t) \equiv \Xi(t)$*, and (B) Conjecture 9.1 closes, giving* $\Delta(t) \equiv \Xi_{\mathrm{APM}}(t)$ *as a canonically normalized operator determinant of a self-adjoint compact (trace-class) family.*

Then the classical Riemann Ξ arises as an operator determinant of a self-adjoint compact family on finite windows, with a controlled window-removal and cutoff-removal limit. The remaining RH-critical step is to supply a canonical positivity mechanism (or equivalent spectral constraint) that forces all zeros of $\Delta(t)$ to occur on the real t-axis, thereby locating the nontrivial zeros on the critical line.

This places RH resolution squarely in the operator-positivity / canonical-system domain, aligned with the Hilbert–Pólya philosophy while remaining within the APM-derived operator framework.

Remark 10.2 (Statistics checkpoint). As an intermediate diagnostic, the zero statistics extracted from stabilized determinant proxies should be compared against known pair-correlation / random-matrix expectations Rudnick and Sarnak [1996], Berry and Keating [1999]. This serves as a falsification lever for inadmissible deformations of the kernel or closure rules, not as an axiom.

11 Computational protocol (determinant generator scan)

1. Fix (q_-, q_+) and holonomy phase ϑ, hence L_q and the scanner grid t_n.
2. Choose an APM-admissible $V_{\mathrm{APM}}(q)$ (including $V \equiv 0$ as calibration).
3. Choose a cutoff E_c and compute an orthonormal basis of $\mathrm{Ran}(\Pi_{E_c})$.
4. Assemble the finite matrix for $S_{E_c}(t)$:

$$S_{jk}(t) = \int_{I\times I} \overline{\psi_j(q)}\, e^{it(q-u)}\, \Phi_{\mathrm{APM}}(q-u)\, \psi_k(u)\, dq\, du.$$

5. Compute $D_{E_c}(t) = \det(S_{E_c}(t))$ (or the renormalized $\det(I + S_{E_c}(t))$) across a real t grid, and locate near-zeros.
6. Increase L_q and verify that the diagonal eigenvalues converge to $\Xi_{\mathrm{APM}}(t+t_n)$ in the calibration regime (Corollary 7.2).
7. Increase E_c and test stability of determinant zeros under admissible perturbations of V_{APM}, using Package-3 stability bounds Thomson III [2026c].

Bornemann's Nyström-type quadrature methods provide a standard high-accuracy pathway for determinant evaluation of compact integral operators and directly inform error control in the determinant scan Bornemann [2010].

A Determinants and trace ideals: minimal background

For a trace-class operator K on a Hilbert space, the Fredholm determinant is defined by

$$\det(I + K) := \prod_j (1 + \lambda_j),$$

where $\{\lambda_j\}$ are eigenvalues of K counted with algebraic multiplicity; equivalently,

$$\log\det(I + K) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \mathrm{Tr}(K^m),$$

with absolute convergence in trace norm. Standard conditions, regularized determinants, and stability bounds are developed systematically in Simon [2005].

In Package 5, $S_{E_c}(t)$ is finite-rank, so $\det(I + S_{E_c}(t))$ is an ordinary determinant while still admitting a trace-ideal limit program as $E_c \to \infty$.

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