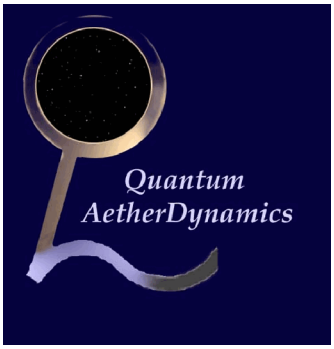

Quantum AetherDynamics Institute (QADI)

Riemann Hypothesis Proof Program: Identification, Determinant Closure, and Positivity Fork for the Riemann Function Operators Series

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It develops the Aether Physics Model (APM)

Abstract

Operator Packages 1–5 develop a self-adjoint, finite-window, log-scale operator framework and a determinant generator whose near-zeros empirically target the zero set of $\Xi(t) = \xi(\frac{1}{2} + it)$. This paper rewrites the series endpoint as a modular proof program. The program is split into three separable modules: (A) an identification module asserting $\Xi_{\mathrm{APM}} \equiv \Xi$ as entire functions, (B) a determinant-closure module asserting that a canonically renormalized Fredholm (or regularized) determinant $\Delta(t)$ exists in the large-limit and equals $\Xi_{\mathrm{APM}}(t)$, and (C) a positivity/canonical-system module forcing Δ to have only real zeros. Each module is expanded into explicit definitions, sufficient-condition theorems, lemma checklists, and falsifiable checkpoints. Under (A)–(C), the Riemann Hypothesis follows immediately.

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1 Scope, RH reformulation, and the program fork

1.1 RH in the xi-variable

Let

$$\Xi(t) := \xi\Big(\tfrac{1}{2} + it\Big), \qquad t \in \mathbb{C},$$

where ξ is the completed zeta function. The Riemann Hypothesis (RH) is equivalent to:

All zeros of $\Xi(t)$ lie on the real axis.

1.2 What Packages 1–5 already supply

The operator series supplies [Thomson III, 2026a,b,c,d,e]:

- a self-adjoint log-translation generator on a finite log-window with quasi-periodic boundary closure (discrete spectrum),
- compact self-adjoint window filters built from an even decaying kernel (Hilbert–Schmidt),

- coherence selection via spectral projections of a self-adjoint Hamiltonian,
- a determinant generator built from finite-rank compressions and a trace-class/regularized renormalization strategy to stabilize rank growth.

1.3 The explicit fork

The series isolates RH resolution into three decisive statements, each independently attackable:

(A) Identification: $\Xi_{\mathrm{APM}} \equiv \Xi$.
(B) Canonical determinant closure: $\Delta \equiv \Xi_{\mathrm{APM}}$.
(C) Positivity/canonical-system forcing real zeros of Δ.

This paper expands (A)–(C) into lemma chains and checklists suitable for a proof-track companion paper sequence.

2 Collected operator definitions (from Packages 1–5)

2.1 Hilbert spaces and finite log-window

Let $H := L^2(\mathbb{R}, \mathrm{d}q)$ and for a finite log-window $I = [q_-, q_+]$ with $L_q = q_+ - q_- > 0$ let $H_I := L^2(I, \mathrm{d}q)$.

2.2 Self-adjoint log-translation generator with holonomy

On I, fix $\vartheta \in \mathbb{R}$ and define

$$(P_\vartheta f)(q) := -i\frac{df}{dq}, \qquad \mathrm{Dom}(P_\vartheta) := \{f \in H^1(I) : f(q_+) = e^{i\vartheta} f(q_-)\}.$$

Then P_ϑ is self-adjoint on H_I with discrete eigenvalues

$$t_n = \frac{2\pi n + \vartheta}{L_q}, \qquad n \in \mathbb{Z},$$

and corresponding Fourier basis on the window.

2.3 Ledger normalization (QMU anchor used in the series)

Define the ledger propagation constant

$$\kappa_0 := F_q \, \lambda_C,$$

with the series identity $\kappa_0^2 = A_u \cdot \mathrm{curl}$ (Ledger One closure). The ledger-scaled generator is

$$P_L := \kappa_0 \, P_\vartheta.$$

(Only the algebraic role of κ_0 is used in this RH paper; no external unit system is invoked.)

2.4 Kernel, window filter, and log-phase twist

Let $\Phi_{\mathrm{APM}} : \mathbb{R} \to \mathbb{R}$ be an even decaying kernel. Define the window filter on H_I by

$$(W_I f)(q) := \int_{q_-}^{q_+} \Phi_{\mathrm{APM}}(q-u)\, f(u)\, \mathrm{d}u.$$

Assumptions used throughout:

(K1) $\Phi_{\mathrm{APM}} \in L^2(\mathbb{R})$ and is even: $\Phi_{\mathrm{APM}}(u) = \Phi_{\mathrm{APM}}(-u)$.
(K2) Φ_{APM} is real-valued (hence W_I is self-adjoint).

Then W_I is Hilbert–Schmidt (hence compact) and self-adjoint on H_I.

Let $(Qf)(q) = qf(q)$ and define the unitary log-phase twist

$$(U_t f)(q) := e^{itq} f(q), \qquad t \in \mathbb{R}.$$

The twisted window filter is

$$W_I(t) := U_t W_I U_t^{-1},$$

with kernel $K_t(q,u) = e^{it(q-u)}\Phi_{\mathrm{APM}}(q-u)$ on $I \times I$.

2.5 Coherence selection and finite-rank compressions

Let H_{APM} be a self-adjoint Sturm–Liouville Hamiltonian on H_I with bounded real potential V_{APM}. For an energy cutoff E_c, let

$$\Pi_{E_c} := \mathbf{1}_{(-\infty, E_c]}(H_{\mathrm{APM}})$$

be the spectral projector. Define the finite-rank coherence-selected operator

$$S_{E_c,I}(t) := \Pi_{E_c} W_I(t) \Pi_{E_c}.$$

Then $S_{E_c,I}(t)$ is finite rank and self-adjoint for real t.

2.6 Determinant generator (finite-rank) and canonical stabilization

On $\mathrm{Ran}(\Pi_{E_c})$ define

$$D_{E_c,I}(t) := \det(S_{E_c,I}(t)).$$

To pass to large limits ($E_c \to \infty$, $L_q \to \infty$) one must stabilize rank growth. This paper allows two canonical determinant routes:

(B-route 1) Trace-class route: construct a canonically normalized *trace-class* family $K_{E_c,I}(t)$ and use $\det(I+K)$.
(B-route 2) Hilbert–Schmidt route: construct a canonically normalized *Hilbert–Schmidt* family $K_{E_c,I}(t)$ and use the regularized determinant $\det_2(I+K)$.

Both routes appear in standard Fredholm determinant numerics and theory [Bornemann, 2010, Simon, 2005].

3 Module A: Identification $\Xi_{\mathrm{APM}} \equiv \Xi$

3.1 Definition of Ξ_{APM}

Fix a Fourier convention. For $\Phi \in L^1(\mathbb{R})$ define

$$\widehat{\Phi}(t) := \int_{\mathbb{R}} \Phi(u)\, e^{itu}\, \mathrm{d}u, \qquad t \in \mathbb{R}.$$

Assume additionally:

(K3) $\Phi_{\mathrm{APM}} \in L^1(\mathbb{R})$ and $\widehat{\Phi_{\mathrm{APM}}}$ extends to an entire function on $\mathbb{C}$.

Then define the APM multiplier

$$\Xi_{\mathrm{APM}}(t) := \widehat{\Phi_{\mathrm{APM}}}(t), \qquad t \in \mathbb{C}.$$

3.2 Uniqueness lemmas (low-cost, high-value)

Lemma 3.1 (Fourier uniqueness on $\mathbb{R}$)**.** *If $\Phi_1, \Phi_2 \in L^1(\mathbb{R})$ and $\widehat{\Phi_1}(t) = \widehat{\Phi_2}(t)$ for all $t \in \mathbb{R}$, then $\Phi_1 = \Phi_2$ almost everywhere.*

Lemma 3.2 (Analytic continuation uniqueness)**.** *If F, G are entire and $F(t) = G(t)$ for all $t \in \mathbb{R}$, then $F \equiv G$ on $\mathbb{C}$.*

3.3 Statement A (expanded) and two sufficient-condition routes

Theorem 3.3 (A: Identification target)**.** *Assume Φ_{APM} satisfies (K1)–(K3) and that for real t,*

$$\Xi_{\mathrm{APM}}(t) = \Xi(t).$$

Then $\Xi_{\mathrm{APM}} \equiv \Xi$ as entire functions.

Proof. By Definition 3.2, equality on $\mathbb{R}$ implies equality on $\mathbb{C}$. □

The program requirement is thus reduced to establishing $\Xi_{\mathrm{APM}}(t) = \Xi(t)$ for real t. Two sufficient-condition routes are recorded as theorem targets.

Route A-I (structural/theta route). Prove that the APM construction produces a kernel representation equivalent to a known xi-kernel representation. In practice this typically means:

1. a partition kernel Θ_{APM} satisfying a modular inversion symmetry,
2. a normalization matching the classical completed symmetry,
3. an explicit derivation showing Φ_{APM} matches (or is uniquely determined by) the classical xi-kernel.

This route usually produces direct equality $\Xi_{\mathrm{APM}}(t) = \Xi(t)$ on $\mathbb{R}$.

Route A-II (Maclaurin jet + growth route). Prove:

1. Ξ_{APM} is even and real on $\mathbb{R}$ (automatic under (K1)–(K2)),
2. Ξ_{APM} is entire of order 1 with matching type constraints,
3. the even derivatives at 0 match: $\Xi_{\mathrm{APM}}^{(2k)}(0) = \Xi^{(2k)}(0)$ for all $k \geq 0$ (or for an infinite uniqueness set with a limit point).

Then $\Xi_{\mathrm{APM}} \equiv \Xi$.

3.4 Checkpoint A (falsifiable)

(A1) *Maclaurin jet test:* compute $\Xi_{\mathrm{APM}}^{(2k)}(0)$ from kernel moments

$$\Xi_{\mathrm{APM}}^{(2k)}(0) = i^{2k} \int_{\mathbb{R}} u^{2k} \Phi_{\mathrm{APM}}(u)\, \mathrm{d}u = (-1)^k \int_{\mathbb{R}} u^{2k} \Phi_{\mathrm{APM}}(u)\, \mathrm{d}u,$$

and verify agreement with the corresponding xi-derivatives for $k = 0, \ldots, N$.

(A2) *Order/type test:* verify the growth class of Ξ_{APM} matches the known order 1 behavior of Ξ.

4 Module B: Canonical determinant closure $\Delta \equiv \Xi_{\mathrm{APM}}$

4.1 Determinant class options

Let K be a compact operator.

- If K is trace class, define the Fredholm determinant

$$\det(I + K) := \prod_j (1 + \lambda_j),$$

where $\{\lambda_j\}$ are eigenvalues of K counted with algebraic multiplicity.

- If K is Hilbert–Schmidt, define the regularized determinant

$$\det_2(I + K) := \det\big((I + K)e^{-K}\big).$$

Both determinants admit trace expansions under their respective norm controls [Bornemann, 2010, Simon, 2005].

4.2 A canonical closure target (explicit)

The goal is to produce a canonically normalized family $K_{E_c,I}(t)$ from the coherence-selected operators (or an equivalent renormalized family) such that:

(B1) For each real t, $K_{E_c,I}(t)$ is self-adjoint and belongs to the determinant class used (trace class or Hilbert–Schmidt).

(B2) The map $t \mapsto K_{E_c,I}(t)$ is analytic (operator-norm analytic on compact sets is sufficient).

(B3) There exists a large-limit operator $K(t)$ with convergence $K_{E_c,I}(t) \to K(t)$ in the appropriate determinant topology as $E_c \to \infty$ and $L_q \to \infty$.

(B4) The determinant

$$\Delta(t) := \begin{cases} \det(I + K(t)), & \text{trace-class route,} \\ \det_2(I + K(t)) \cdot e^{c(t)}, & \text{HS route with canonical exponential correction,} \end{cases}$$

is an entire function and satisfies $\Delta(t) \equiv \Xi_{\mathrm{APM}}(t)$.

The HS route includes an explicit scalar correction $c(t)$ determined canonically from the renormalization; its role is to eliminate spurious entire prefactors.

4.3 Continuity lemmas to be used

Lemma 4.1 (Determinant continuity (trace class)). *If $K_n \to K$ in trace norm, then $\det(I + K_n) \to \det(I + K)$ and the convergence is locally uniform in any analytic parameter on which K_n depends continuously in trace norm.*

Lemma 4.2 (Determinant continuity (Hilbert–Schmidt)). *If $K_n \to K$ in Hilbert–Schmidt norm, then $\det_2(I + K_n) \to \det_2(I + K)$ locally uniformly in analytic parameters.*

Lemma 4.3 (Log-derivative trace identity (formal target)). *Assume $t \mapsto K(t)$ is differentiable in trace norm and $I + K(t)$ is invertible. Then*

$$\frac{d}{dt} \log \det(I + K(t)) = \mathrm{Tr}\Big((I + K(t))^{-1} K'(t)\Big).$$

An analogous identity holds for $\det_2$ with the standard correction term.

4.4 Statement B (expanded)

Conjecture 4.4 (B: Canonical determinant closure target). There exists a canonically normalized family $K_{E_c,I}(t)$, built from the Package-5 renormalization of coherence-selected compressions, and a limiting family $K(t)$ such that:

1. $K(t)$ is in the determinant class (trace class or Hilbert–Schmidt) for all real t and depends analytically on t,
2. $\Delta(t)$ defined from $K(t)$ is entire,
3. and the identity
$$\Delta(t) \equiv \Xi_{\mathrm{APM}}(t)$$
holds as entire functions.

4.5 Checkpoint B (falsifiable)

(B1) *Rank stability:* computed near-zeros of finite-rank determinants (or their stabilized analogs) persist under increasing E_c.

(B2) *Window stability:* the same near-zeros persist under increasing L_q.

(B3) *Normalization check:* enforce $\Delta(0) = \Xi_{\mathrm{APM}}(0)$ and verify that Δ'/Δ matches the trace identity prediction from Definition 4.3 on test intervals.

5 Module C: Positivity / canonical-system forcing real zeros

5.1 Three sufficient-condition routes (choose one as primary)

Module C is the RH-critical step: it forces the limiting entire function Δ to have no nonreal zeros. This paper records three sufficient-condition theorems; proving any one completes C.

5.2 Route C2 (primary target here): Herglotz log-derivative

Define for $z \in \mathbb{C}$ off the zeros of Δ,

$$m(z) := -\frac{\Delta'(z)}{\Delta(z)}.$$

Definition 5.1 (Herglotz function)**.** A function m is Herglotz (Nevanlinna) if it is analytic on $\{z : \Im z > 0\}$ and satisfies $\Im m(z) > 0$ whenever $\Im z > 0$.

Theorem 5.2 (C2: Herglotz implies real zeros)**.** *If $m(z) = -\Delta'(z)/\Delta(z)$ is Herglotz, then Δ has no zeros in $\Im z \neq 0$. Equivalently, all zeros of Δ lie on the real axis.*

5.3 Route C1: de Branges / canonical systems (alternate)

Theorem 5.3 (C1: de Branges sufficiency (target))**.** *If Δ is (up to an explicit nonvanishing entire factor) the de Branges function associated with a canonical system whose Hamiltonian is positive semidefinite almost everywhere, then all zeros of Δ lie on the real axis.*

5.4 Route C3: Laguerre–Pólya via real-zero approximation (alternate)

Theorem 5.4 (C3: LP class via real-zero limits (target))**.** *Assume there exist polynomials (or finite products) Δ_n with only real zeros such that $\Delta_n \to \Delta$ locally uniformly on $\mathbb{C}$. Then Δ lies in the Laguerre–Pólya class and has only real zeros.*

5.5 Statement C (expanded as a checklist)

Theorem 5.5 (C: Positivity/canonical-system target (sufficient form))**.** *Assume* any one *of Definitions 5.2 to 5.4 holds for the determinant-closed function Δ. Then all zeros of Δ lie on the real axis.*

5.6 Checkpoint C (falsifiable)

(C1) *Nonreal zero stability test:* if a nonreal zero of the stabilized determinant proxy persists under increasing E_c and L_q, the chosen positivity route fails and RH cannot be concluded by this closure.

(C2) *Herglotz diagnostic (if using C2):* sample $\Im m(z)$ numerically for $\Im z > 0$ using stable evaluations of Δ and verify sign consistency on a grid.

6 Program closure theorem

Theorem 6.1 (RH closure under Modules A–C)**.** *Assume:*

1. *Module A holds:* $\Xi_{\mathrm{APM}} \equiv \Xi$.
2. *Module B holds:* $\Delta \equiv \Xi_{\mathrm{APM}}$.
3. *Module C holds: all zeros of* Δ *lie on the real axis.*

Then RH holds.

Proof. By A and B, $\Delta \equiv \Xi$ as entire functions. By C, all zeros of Δ lie on the real axis, hence all zeros of Ξ lie on the real axis. This is equivalent to RH. □

7 Verification protocol (one-page operational summary)

7.1 Protocol overview

The program is intentionally falsifiable. The protocol is to test A, then B, then C, in that order.

7.2 Protocol A: identification

1. Verify kernel hypotheses (K1)–(K3).
2. Run Checkpoint (A1) for increasing jet depth N.
3. Run Checkpoint (A2) to constrain growth class.

7.3 Protocol B: closure

1. Construct stabilized determinant proxies using the Package-5 renormalization.
2. Run Checkpoints (B1)–(B2) for increasing E_c and L_q.
3. Enforce normalization and trace-identity diagnostics (B3).

7.4 Protocol C: positivity

1. Choose a route (primary: C2).
2. If C2, evaluate $m(z)$ and test $\Im m(z) > 0$ on $\Im z > 0$ grids (C2).
3. Regardless of route, attempt nonreal-zero stability falsification (C1).

A Lemma inventory (proof delegation list)

This appendix lists the minimal lemma set required for a complete proof under each module. Each item is phrased so it can become a stand-alone lemma in the next paper of the series.

A-lemmas (identification)

(A-L1) Prove (K3): entire extension of Ξ_{APM} from the APM kernel construction.

(A-L2) Prove a structural derivation that $\Xi_{\mathrm{APM}}(t) = \Xi(t)$ for all real t (Route A-I) *or* prove a uniqueness-set match (Route A-II).

B-lemmas (determinant closure)

(B-L1) Construct the canonically renormalized $K_{E_c,I}(t)$ and prove determinant-class membership (trace class or HS) uniformly on compact t-sets.

(B-L2) Prove convergence $K_{E_c,I}(t) \to K(t)$ in the determinant topology as $E_c \to \infty$.

(B-L3) Prove window removal $L_q \to \infty$ does not introduce a spurious entire prefactor (canonicality).

(B-L4) Prove the log-derivative identity Definition 4.3 holds for the limiting family and yields $\Delta \equiv \Xi_{\mathrm{APM}}$ after normalization at $t = 0$.

C-lemmas (positivity)

(C-L1) (C2 route) Represent $m(z)$ as a resolvent trace of a self-adjoint operator family to obtain the Herglotz property.

(C-L2) (C1 route) Build the canonical system and prove Hamiltonian positivity.

(C-L3) (C3 route) Prove each finite-stage proxy has only real zeros and converges locally uniformly to Δ.

B Concluding consolidation

Packages 1–5 already provide an operator scaffold and a determinant generator with empirical targeting of xi-zeros. The remaining work is cleanly modular: (A) identify Ξ_{APM} with Ξ, (B) close the stabilized determinant to Ξ_{APM} in the large-limit, (C) force real zeros by a positivity/canonical-system mechanism. This rewrite turns that strategy into an explicit theorem-and-lemma checklist with falsifiable checkpoints.

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